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Liouville field theory on an unoriented surface

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Abstract: Liouville field theory on an unoriented surface is investigated, in particular, the one point function on a $\mathbb{RP}^2$ is calculated. The constraint of the one point function is obtained by using the crossing symmetry of the two point function. There are many solutions of the constraint and we can choose one of them by considering the modular bootstrap.

Keywords: D-branes, Conformal Field Models in String Theory.

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Contents

1. Introduction

In the recent development of the string theory, the D-branes become the crucial objects. Among other things, they are used for the investigation of the string duality and AdS/CFT correspondence. However, it is known that there are tadpoles in the configurations only with D-branes and we have to introduce the orientifolds in order to cancel the tadpoles. Thus, it is also important to investigate the orientifolds.

The most famous example is the type I string theory, which can be regarded as the type IIB string theory with (1+9) dimensional orientifold plane. This theory is defined on unoriented worldsheets with and without boundary. Although we know well the orientifolds in the flat space, we have little knowledge about the orientifolds in the curved backgrounds. Only in the case of rational conformal field theory, the orientifolds have been investigated [1]–[3] and their geometrical pictures are given recently [4]–[9].

In this paper, we consider the Liouville field theory on an unoriented surface as the simple example of non-rational case. Liouville field theory is also interesting because it appears in several important systems. This theory was much investigated about ten years ago because of the relation with the two dimensional quantum gravity. It is known that the Liouville field theory is dual to the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ WZW model, which appears in superstring theory as an interesting solvable case. In addition, the AdS_3 string theory is resemble to the Liouville field theory, thus it is important in a sense of the AdS/CFT correspondence.

The Liouville field theory with boundary is studied in [10]–[16] and we will follow their analysis. First we obtain the solutions of the one point function by making use of the crossing symmetry on the two point function. Then we determine the precise form by considering the one loop partition function. The D-branes in the AdS_3 space are much

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investigated [17]–[33] in the similar manner. The orientifold of the AdS_3 space is also constructed in [34], however the constraint is too weak and we cannot determine the precise form of the one point functions.[1] It is a better point that we can determine the exact form in the Liouville field theory case.

The organization of this paper is as follows. In section 2, we review the Liouville field theory on a sphere and summarize our notations. In section 3, the one point function on a $\mathbb{RP}^2$ is examined. We obtain the constraints from the crossing symmetry of the two point functions and then we solve these constraints. In section 4, we consider the modular bootstrap and the precise form of the one point function is determined. The conclusion and discussions are given in section 5 and the several useful formulae are summarized in appendix A.

2. Liouville field theory

The Liouville Field Theory is defined by the action

$$S = \frac{1}{4\pi} \int d^2x \sqrt{g} \left[g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + QR\phi + 4\pi\mu e^{2b\phi} \right], \tag{2.1}$$

where g is the metric and R is the scalar curvature. The quantity $Q = b + 1/b$ is called as the background charge and μ is called as the cosmological constant. By setting $\mu = 0$, the stress tensors are given by

$$T(z) = -(\partial\phi)^2 + Q\partial^2\phi\,, \qquad \bar{T}(\bar{z}) = -(\bar{\partial}\phi)^2 + Q\bar{\partial}^2\phi\,, \tag{2.2}$$

and the central charge of the theory is $c = 1 + 6Q^2$. The primary fields are defined as $V_\alpha = \exp(2\alpha\phi(x))$ with the conformal weights $\Delta_\alpha = \alpha(Q - \alpha)$. The normalizable states correspond to the operators with $\alpha = Q/2 + iP$, where we restrict $P \geq 0$ since the operators V_α and $V_{Q-\alpha}$ are related by so called reflection relations [36].

We can investigate the conformal field theory by considering the correlation functions of the fields. In principle, the multi-point correlation functions can be calculated from the information of the two point functions and three point functions:

$$\begin{aligned} \langle V_\alpha(x) V_\alpha(y) \rangle &= \frac{D(\alpha)}{|x-y|^{4\Delta_\alpha}}\,, \\ \langle V_{\alpha_1}(x_1) V_{\alpha_2}(x_2) V_{\alpha_3}(x_3) \rangle &= \frac{C(\alpha_1, \alpha_2, \alpha_3)}{|x_1 - x_2|^{2\Delta_{12}} |x_2 - x_3|^{2\Delta_{23}} |x_3 - x_1|^{2\Delta_{31}}}\,, \end{aligned} \tag{2.3}$$

where we use

$$\Delta_{12} = \Delta_{\alpha_1} + \Delta_{\alpha_2} - \Delta_{\alpha_3}\,, \qquad \Delta_{23} = \Delta_{\alpha_2} + \Delta_{\alpha_3} - \Delta_{\alpha_1}\,, \qquad \Delta_{31} = \Delta_{\alpha_3} + \Delta_{\alpha_1} - \Delta_{\alpha_2}\,. \tag{2.4}$$

These quantities can be obtained by using the following technique. Among the general states, there are special states which are degenerate

$$\Phi_{m,n} = e^{\left((1-m)\frac{1}{b} + (1-n)b\right)\phi}\,, \tag{2.5}$$

[1] The one point function can be determined up to overall factor with the help of the geometric interpretation [34].

and they satisfy some differential equations. The simplest one is given for $\Phi_{1,2} = V_{-b/2}$ as

$$\left(\frac{1}{b^2}\partial^2 + T(z)\right) V_{-b/2} = 0\,. \tag{2.6}$$

When considering the operator product expansions including the degenerate states, these differential equations restrict the number of primary fields. For the above example $\Phi_{1,2} = V_{-b/2}$, we find

$$V_{-b/2}V_\alpha \sim C_+ V_{\alpha-b/2} + C_- V_{\alpha+b/2}\,, \tag{2.7}$$

where the coefficients can be calculated as $C_+ = 1$ and

$$C_- = -\mu\pi \frac{\gamma(2b\alpha - 1 - b^2)}{\gamma(-b^2)\gamma(2b\alpha)}\,, \qquad \gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}\,. \tag{2.8}$$

Using this operator product expansion, we can evaluate the three point function including this field in two ways. Equating two quantities, we obtain the constraint and the solution is given by

$$D(\alpha) = \frac{1}{b^2}(\pi\mu\gamma(b^2))^{(Q-2\alpha)/b} \frac{\gamma(2b\alpha - b^2)}{\gamma(2 - 2\alpha/b + 1/b^2)}\,. \tag{2.9}$$

The constraint does not determine the unique solution, however there is a quite strong constraint that the quantities obtained should be related by the duality $b \leftrightarrow 1/b$. This duality should be understood by also replacing the cosmological constant μ with $\tilde{\mu}$ satisfying

$$\pi\tilde{\mu}\gamma\left(\frac{1}{b^2}\right) = \left(\pi\mu\gamma(b^2)\right)^{1/b^2}\,. \tag{2.10}$$

The general three point functions can be evaluated in the similar way and their explicit forms are obtained in [35]–[39].

3. One point function on a $\mathbb{RP}^2$

For the oriented surface with boundary, there are several constraints of the theory which are called as the sewing constraints [40], and for the unoriented surface, there are three types of additional constraints [41]. In this section, we use the constraint related to the one point functions on the $\mathbb{RP}^2$ and in the next section we see the other two types of constraints from Möbius strip and Klein bottle amplitudes.

The one point function on a $\mathbb{RP}^2$ can be calculated by using the mirror technique. In the case of the one point function on a disk, we can use the upper half plane by conformal mapping from the disk. Then, we can map from the upper half plane to the whole plane by using the involution $I(z) = \bar{z}$. There is a fixed line $\operatorname{Im} z = 0$, which corresponds to the boundary. In the case of the one point function on a $\mathbb{RP}^2$, we can also use the upper half plane, however we should use other involution $I(z) = -1/\bar{z}$ and there is no boundary. By using these mirror techniques, the one point function on a $\mathbb{RP}^2$ can be written as

$$\langle V_\alpha(z,\bar{z})\rangle_{\mathbb{RP}^2} = \frac{U(\alpha)}{|1 + z\bar{z}|^{2\Delta_\alpha}}\,, \tag{3.1}$$

where the z dependence is determined by the conformal symmetry.

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In order to determine the coefficient $U(\alpha)$, we use the two point function including the degenerate field $V_{-b/2}$ just like the bulk case as

$$\langle V_{-b/2}(z,\bar{z})V_\alpha(w,\bar{w})\rangle_{\mathbb{RP}^2}\,. \tag{3.2}$$

When two points z and w are close, it is natural to use the OPE (2.7) and we can write as

$$\begin{aligned}\langle V_{-\frac{b}{2}}(z,\bar{z})V_\alpha(w,\bar{w})\rangle_{\mathbb{RP}^2} &= \frac{|1+w\bar{w}|^{2\Delta_\alpha-2\Delta_{-b/2}}}{|1+z\bar{w}|^{4\Delta_\alpha}}\times \\ &\times\left(C_+(\alpha)U\left(\alpha-\frac{b}{2}\right)\mathcal{F}_+(\eta)+C_-(\alpha)U\left(\alpha+\frac{b}{2}\right)\mathcal{F}_-(\eta)\right),\end{aligned} \tag{3.3}$$

where we define the cross ratio as

$$\eta=\frac{|z-w|^2}{(1+z\bar{z})(1+w\bar{w})}\,. \tag{3.4}$$

Since this correlation function includes the degenerate field $\Phi_{1,2}=V_{-b/2}$ which satisfies (2.6), the conformal blocks $\mathcal{F}_\pm$ also satisfy the differential equation and they can be obtained by solving the differential equation. These solutions are expressed by the hypergeometric functions as

$$\begin{aligned}\mathcal{F}_+(\eta) &= \eta^{b\alpha}(1-\eta)^{b\alpha}F(-1+2b\alpha-2b^2,2b\alpha,2b\alpha-b^2;\eta)\,,\\ \mathcal{F}_-(\eta) &= \eta^{1-b\alpha+b^2}(1-\eta)^{b\alpha}F(1+b^2,-b^2,2-2b\alpha+b^2;\eta)\,.\end{aligned} \tag{3.5}$$

Some properties of the hypergeometric functions are summarized in appendix A.

We should notice that the involution I acts to the field as

$$I: V_\alpha(z,\bar{z})\to\epsilon_\alpha V_\alpha\left(-\frac{1}{\bar{z}},-\frac{1}{z}\right), \tag{3.6}$$

where the phase factor should be $\epsilon_\alpha=\pm1$ since the product of two involutions is the identity. In the rational conformal field theory case, the label of fields takes a discrete number, therefore we can choose an arbitrary sign for the different fields as long as they are consistent with the OPE. On the other hand, the label of fields in our case takes the continuous value. Thus we can see that the consistency of OPE implies $\epsilon=+1$. By using this fact, we find

$$\langle V_\alpha(z,\bar{z})\cdots\rangle=\left\langle V_\alpha\left(-\frac{1}{\bar{z}},-\frac{1}{z}\right)\cdots\right\rangle, \tag{3.7}$$

and this equation gives constraint to the one point function [41].

The following two point function can be calculated in the similar way as

$$\begin{aligned}&\left\langle V_{-b/2}(z,\bar{z})V_\alpha\left(-\frac{1}{\bar{w}},-\frac{1}{w}\right)\right\rangle_{\mathbb{RP}^2}= \\ &=\frac{|1+w\bar{w}|^{2\Delta_\alpha-2\Delta_{-b/2}}}{|1+z\bar{w}|^{4\Delta_\alpha}}\left(C_+(\alpha)U\left(\alpha-\frac{b}{2}\right)\mathcal{F}_+(1-\eta)+C_-(\alpha)U\left(\alpha+\frac{b}{2}\right)\mathcal{F}_-(1-\eta)\right).\end{aligned} \tag{3.8}$$

Using the properties of the hypergeometric functions in appendix A, we obtain

$$\begin{aligned}
\mathcal{F}_+(1-\eta) &= \frac{\Gamma(2b\alpha-b^2)\Gamma(1-2b\alpha+b^2)}{\Gamma(1+b^2)\Gamma(-b^2)}\mathcal{F}_+(\eta) + \\
&\quad + \frac{\Gamma(2b\alpha-b^2)\Gamma(-1+2b\alpha-b^2)}{\Gamma(-1+2b\alpha-2b^2)\Gamma(2\alpha b)}\mathcal{F}_-(\eta)\,, \\
\mathcal{F}_-(1-\eta) &= \frac{\Gamma(2-2b\alpha+b^2)\Gamma(1-2b\alpha+b^2)}{\Gamma(1-2b\alpha)\Gamma(2-2\alpha b+2b^2)}\mathcal{F}_+(\eta) + \\
&\quad + \frac{\Gamma(2-2b\alpha+b^2)\Gamma(-1+2b\alpha-b^2)}{\Gamma(1+b^2)\Gamma(-b^2)}\mathcal{F}_-(\eta)\,.
\end{aligned} \tag{3.9}$$

Now we can compare two quantities (3.3) and (3.8) by using (3.7). Then the following constraints are obtained as

$$\begin{aligned}
U\left(\alpha-\frac{b}{2}\right) &= \frac{\Gamma(2b\alpha-b^2)\Gamma(1-2b\alpha+b^2)}{\Gamma(1+b^2)\Gamma(-b^2)}U\left(\alpha-\frac{b}{2}\right) - \\
&\quad - \frac{\pi\mu}{\gamma(-b^2)}\frac{\Gamma(1-2b\alpha+b^2)\Gamma(-1+2b\alpha-b^2)}{\Gamma(2b\alpha)\Gamma(2-2b\alpha+2b^2)}U\left(\alpha+\frac{b}{2}\right), \\
U\left(\alpha+\frac{b}{2}\right) &= \frac{\Gamma(2-2b\alpha+b^2)\Gamma(-1+2b\alpha-b^2)}{\Gamma(1+b^2)\Gamma(-b^2)}U\left(\alpha+\frac{b}{2}\right) - \\
&\quad - \frac{\gamma(-b^2)}{\pi\mu}\frac{\Gamma(2-2b\alpha+b^2)\Gamma(2b\alpha-b^2)}{\Gamma(1-2b\alpha)\Gamma(-1+2b\alpha-2b^2)}U\left(\alpha-\frac{b}{2}\right).
\end{aligned} \tag{3.10}$$

These constraints can be solved and the solutions are of the forms as

$$U(\alpha) = \frac{2}{b}\left(\pi\mu\gamma(b^2)\right)^{(Q-2\alpha)/2b}\Gamma(2b\alpha-b^2)\Gamma\left(\frac{2\alpha}{b}-\frac{1}{b^2}-1\right)f(\alpha)\,. \tag{3.11}$$

The function $f(\alpha)$ is given by the linear combination of the following two functions as

$$\cos\left(\left(b+\frac{1}{b}\right)\pi\left(\alpha-\frac{Q}{2}\right)\right), \qquad \cos\left(\left(b-\frac{1}{b}\right)\pi\left(\alpha-\frac{Q}{2}\right)\right), \tag{3.12}$$

where we have used the duality $b \leftrightarrow 1/b$ in order to restrict the form of the solutions. We should notice that these solutions satisfy the reflection relation [36]

$$U(\alpha) = U(Q-\alpha)D(\alpha)\,, \tag{3.13}$$

where $D(\alpha)$ is the coefficient of the two point function (2.9). Although we cannot determine the coefficients at this level, they can be fixed by considering the Möbius strip amplitude as we will see in the next section.

4. Crosscap state and modular bootstrap

It is convenient to introduce the boundary states and the crosscap state for considering the Liouville field theory on the annulus, Möbius strip and Klein bottle. First, let us review the analysis of the boundary states [10, 12]. The Virasoro character of the general

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non-degenerate representation $\alpha = Q/2 + iP$ is given by

$$\chi_P(\tau) = \mathrm{Tr}_{\mathcal{H}_P}(q^{L_0 - c/24}) = \frac{q^{P^2}}{\eta(\tau)}\,, \tag{4.1}$$

where the eta function $\eta(\tau)$ is defined in appendix A. The modular transformation can be written as

$$\chi_P\left(-\frac{1}{\tau}\right) = \sqrt{2}\int dP' \chi_{P'}(\tau) e^{4\pi i P P'}\,. \tag{4.2}$$

For the degenerate state $\Phi_{m,n}$, the character is

$$\chi_{m,n}(\tau) = \frac{q^{-\frac{1}{4}\left(\frac{m}{b}+nb\right)^2} - q^{-\frac{1}{4}\left(\frac{m}{b}-nb\right)^2}}{\eta(\tau)}\,, \tag{4.3}$$

which transforms under the modular transformation as

$$\begin{aligned}\chi_{m,n}\left(-\frac{1}{\tau}\right) &= \sqrt{2}\int dP \chi_P(\tau)\left(\cosh\left(2\pi P\left(\frac{m}{b}+nb\right)\right) - \cosh\left(2\pi P\left(\frac{m}{b}-nb\right)\right)\right)\\ &= 2\sqrt{2}\int dP \chi_P(\tau)\sinh\left(\frac{2\pi m P}{b}\right)\sinh(2\pi n b P)\,.\end{aligned} \tag{4.4}$$

The boundary states are described in terms of the Ishibashi states [42] which satisfy

$$_I\langle P| q^{\frac{1}{2}(L_0 + \bar{L}_0 - \frac{c}{12})} |P'\rangle_I = \delta_{P,P'}\chi_P(\tau)\,. \tag{4.5}$$

The general boundary states can be written by the linear combination of the Ishibashi states. The coefficients correspond to the one point functions since they can be calculated by the overlaps between the boundary states and closed string states.

The one point function on a pseudosphere was obtained in [12] and the corresponding boundary states are labeled by (m,n) as

$$_C\langle m,n| = \int dP \Psi_{m,n}(P)\, _I\langle P|\,. \tag{4.6}$$

The boundary state $|1,1\rangle_C$ can be interpreted as a basic state and the wave function $\Psi_{1,1}(P)$ is

$$\Psi_{1,1}(P) = \frac{2^{3/4} 2\pi i P}{\Gamma(1-2ibP)\Gamma(1-2iP/b)}(\pi\mu\gamma(b^2))^{-iP/b}\,. \tag{4.7}$$

The other wave functions $\Psi_{m,n}$ are expressed in this basis as

$$\Psi_{m,n}(P) = \Psi_{1,1}(P)\frac{\sinh(2\pi m P/b)\sinh(2\pi n b P)}{\sinh(2\pi P/b)\sinh(2\pi b P)}\,. \tag{4.8}$$

There are the other kind of boundary states which correspond to the one point functions on the disk [10]. The wave functions can be labeled by a continuous number s and they are given by

$$\Psi_s(P) = \frac{2^{-1/4}\Gamma(1+2ibP)\Gamma(1+2iP/b)\cos(2\pi s P)}{-2i\pi P}(\pi\mu\gamma(b^2))^{-iP/b}\,. \tag{4.9}$$

The normalization of wave functions is determined by using the basic boundary states.

Next, we construct the crosscap state. For the Möbius strip amplitudes, it is convenient to introduce the following characters [1] as

$$\hat{\chi}_\alpha(q) = e^{-\pi i(\Delta_\alpha - c/24)}\chi_\alpha(-\sqrt{q})\,. \tag{4.10}$$

The modular transformation of the Möbius strip can be performed by so called P matrix ($P = \sqrt{T}ST^2S\sqrt{T}$). This matrix transforms $\tau \to -1/(4\tau)$ and for the character of the non-degenerate representation it can be given by

$$\frac{e^{2\pi i(-1/4\tau)P^2}}{\eta\left(-1/4\tau\right)} = \int dP' e^{2\pi i P P'}\frac{e^{2\pi i\tau P'^2}}{\eta(\tau)}\,, \tag{4.11}$$

and for the character of the degenerate representation it can be written as

$$\begin{aligned}
&\frac{e^{-2\pi i\left(-\frac{1}{4\tau}\right)\frac{1}{4}\left(\frac{m}{b}+nb\right)^2} - e^{-2\pi i\left(-\frac{1}{4\tau}\right)\frac{1}{4}\left(\frac{m}{b}-nb\right)^2}}{\eta\left(-\frac{1}{4\tau}\right)} = \\
&= \int dP \frac{e^{2\pi i\tau P^2}}{\eta(\tau)}\left(\cosh\left(\pi P\left(\frac{m}{b}+nb\right)\right) - \cosh\left(\pi P\left(\frac{m}{b}-nb\right)\right)\right) \\
&= 2\int dP \frac{e^{2\pi i\tau P^2}}{\eta(\tau)}\sinh\left(\frac{\pi m P}{b}\right)\sinh(\pi n b P)\,.
\end{aligned} \tag{4.12}$$

In the case of crosscap state, the Ishibashi states are defined by

$$\begin{aligned}
{}_I\langle C, P|q^{\frac{1}{2}(L_0+\bar{L}_0-c/12)}|C, P'\rangle_I &= \delta_{P,P'}\chi_P(\tau)\,, \\
{}_I\langle B, P|q^{\frac{1}{2}(L_0+\bar{L}_0-c/12)}|C, P'\rangle_I &= \delta_{P,P'}\hat{\chi}_P(\tau)\,.
\end{aligned} \tag{4.13}$$

In this basis, the crosscap state is represented as

$${}_C\langle C| = \int dP \Psi_C(P)\,{}_I\langle C, P|\,, \tag{4.14}$$

where $\Psi_C(P)$ is the wave function corresponding to the crosscap state.

In order to determine the wave function $\Psi_C(P)$, we use the character of the identity representation $(m,n) = (1,1)$. The modular transformation is given in (4.12) and it can be interpreted as

$$\hat{\chi}_{1,1}\left(-\frac{1}{\tau}\right) = \int dP \hat{\chi}_P(\tau)\Psi_{1,1}(P)\Psi_C(-P)\,. \tag{4.15}$$

This equation determine the wave function including the normalization factors as

$$\begin{aligned}
\Psi_C(P) = &\frac{2^{-3/4}\Gamma(1+2ibP)\Gamma(1+2iP/b)}{-2i\pi P}(\pi\mu\gamma(b^2))^{-iP/b}\times \\
&\times\left(\cosh\left(\pi P\left(b+\frac{1}{b}\right)\right) - \cosh\left(\pi P\left(b-\frac{1}{b}\right)\right)\right).
\end{aligned} \tag{4.16}$$

This also determines the precise form of the one point function on the $\mathbb{RP}^2$ (3.12).

Because we obtain the precise form of the crosscap state, we can calculate the other partition functions straightforwardly. The overlaps between the boundary states $|m,n\rangle_C$ (4.6) and the crosscap state $|C\rangle_C$ are given by

$$\begin{aligned} Z_{m,n}(\tau) &= \int dP \hat{\chi}_P(\tau) \Psi_{m,n}(P) \Psi_C(-P) \\ &= 2 \int dP \hat{\chi}_P(\tau) \frac{\sinh(2\pi m P/b) \sinh(2\pi n b P) \sinh(\pi P/b) \sinh(\pi b P)}{\sinh(2\pi P/b) \sinh(2\pi b P)} . \end{aligned} \tag{4.17}$$

By using the formula

$$\frac{\sinh(2\pi n b P)}{\sinh(2\pi b P)} = \sum_{l=1-n,3-n,\cdots}^{n-1} e^{2\pi l b P} , \tag{4.18}$$

we find

$$Z_{m,n}(\tau) = \sum_{k=1-m,3-m,\cdots}^{m-1} \sum_{l=1-n,3-n,\cdots}^{n-1} \hat{\chi}_{2k+1,2l+1}\left(-\frac{1}{\tau}\right) . \tag{4.19}$$

We should note that the coefficient of the character of the identity representation is less than one, which means that the crosscap state we have constructed is the irreducible one.

The other type of the Möbius strip amplitudes correspond to the overlaps between the boundary states parametrized by s and the crosscap state as

$$\begin{aligned} Z_s(\tau) &= \int dP \hat{\chi}_P(\tau) \Psi_s(P) \Psi_C(-P) \\ &= \int dP dP' \hat{\chi}_{P'}\left(-\frac{1}{\tau}\right) e^{2\pi i P P'} \Psi_s(P) \Psi_C(-P) \\ &= \int dP' \hat{\chi}_{P'}\left(-\frac{1}{\tau}\right) \rho(P') , \end{aligned} \tag{4.20}$$

where $\rho(P')$ is the density of states. The last one is the Klein bottle amplitude, which is given by

$$\begin{aligned} Z(\tau) &= \int dP \chi_P(\tau) \Psi_C(P) \Psi_C(-P) \\ &= \int dP dP' \chi_{P'}\left(-\frac{1}{\tau}\right) e^{4\pi i P P'} \Psi_C(P) \Psi_C(-P) \\ &= \int dP' \chi_{P'}\left(-\frac{1}{\tau}\right) \rho(P') . \end{aligned} \tag{4.21}$$

In the case of the boundary states, the density of states can be calculated by the other method and we can compare them. It is interesting to compare these densities of states with the ones obtained by other methods if we could also in the case of crosscap state.

5. Conclusion

Liouville field theory on an unoriented surface is investigated. The basic information is given by the one point function on a $\mathbb{RP}^2$. Since it is difficult to calculate in general, we use the trick which was developed for the bulk three point function [37] and for the one

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point function on a disk [10] and on a pseudosphere [12]. The degenerate states satisfy some differential equations, and hence the two point functions including these states are calculable. By assuming the crossing symmetry, we obtain the constraint for the general one point function (3.10). Although there are plenty of solutions of the constraint, we can choose one of them (4.16) by making use of the modular bootstrap.

Since Liouville field theory is a typical example of the non-rational conformal field theory, the application to the other backgrounds, e.g., AdS_3 spaces [34], might be done by using the methods we have used. Apart from the solvable property, Liouville field theory is interesting because it can be embedded into the full superstring theory. For that purpose, we should extend our analysis to the supersymmetric case like [15, 16]. If we can apply to the consistent superstring theory, the orientifolds in a non-trivial background can be constructed and we may see interesting phenomena in the system with the non-trivial orientifolds.

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A. Several useful formulae

The hypergeometric functions have the following properties under the reparametrizations

$$\begin{aligned} F(a,b,c;\eta) &= (1-\eta)^{c-a-b}F(c-a,c-b,c;\eta)\,, \qquad (A.1)\\ F(a,b,c;1-\eta) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}F(a,b,1-c+a+b;\eta)+\\ &\quad +\eta^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}F(c-a,c-b,1+c-a-b;\eta)\,. \qquad (A.2)\end{aligned}$$

We often use the following formulae for Gamma function as

$$\begin{aligned} \Gamma(1+z) &= z\Gamma(z)\,,\\ \Gamma(1-z)\Gamma(z) &= \frac{\pi}{\sin(\pi z)}\,,\\ \Gamma(1+ix)\Gamma(1-ix) &= \frac{\pi x}{\sinh(\pi x)}\,, \qquad (A.3)\end{aligned}$$

where z is an arbitrary complex number and x is a real number.

The Dedekind eta function is defined by

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)\,, \qquad (A.4)$$

where $q=\exp(2\pi i\tau)$ and its modular transformation is given by

$$\eta(\tau+1) = e^{\pi i/12}\eta(\tau)\,, \qquad \eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\,\eta(\tau)\,. \qquad (A.5)$$

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